Postbuckling of an Annulus

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The postbuckling of an annular plate with a free inner edge and a clamped-movable outer edge is investigated. The von Kármán plate equations for an axisymmetric deformation are two coupled nonlinear ordinary differential equations, which are transformed to a nonlinear eigenvalue problem. Numerical solutions are obtained by employing the related initial value problem in conjunction with a numerical method of integration, and by using the method of continuation. The solution yields a complete description of postbuckling loads and stresses. Results reveal the nonlinear behavior of the annulus beyond the linear buckling load. For computing purposes, this computational method proves to be applicable to other nonlinear eigenvalue problems.

Nomenclature

a, b	= outer and inner radii of the annular plate, respectively
h	= thickness of the plate
E	= Young's modulus
ν	= Poisson ratio
q	= uniform compressive thrust at the outer edge
w	= transverse displacement of the unstrained midplane
ψ	= Airy stress function
r, θ, z	= cylindrical coordinates used to describe the undeformed configuration of the plate
N_r, N_θ	= midplane thrust
ξ, χ	= dimensionless space variables
ϕ, p	= dimensionless variables
λ, α	= nondimensional eigenvalue and amplitude parameters, respectively
$\bar{Y}, \bar{Z}, \bar{H}$	= vector functions
$(M), (N)$	= coefficient matrices
$\bar{S}$	= parameter vector
$J_1^{(k)}$	= Fréchet derivative
$\sigma_r^B, \sigma_\theta^B$	= bending stresses
$\sigma_r^M, \sigma_\theta^M$	= membrane stresses
u, v	= nondimensional functions for the transverse displacement and the compressive thrust, respectively

Introduction

MODERN light-weight designs, where lightness and maximum strength are prime requirements, often require a determination of the behavior of thin plates subjected to inplane loads greater than the linear buckling loads. However, the problem of the postbuckling behavior of thin plates belongs to the area of nonlinear mechanics. Because of the complex nature of the differential equations, closed form solutions would be mathematically difficult, and only solutions depending on various approximations have been published. Friedrichs and Stoker[1,2] gave a complete treatment of the post-buckling of a circular simply supported plate by perturbation and power-series methods. These methods were applied by Bodner[3] to the problem for clamped circular plate. Yanowitch[4] showed that the solution of the problem investigated by Friedrichs and Stoker becomes unstable for large values of the thrust loading parameter. Keller and Reiss[5] applied the finite difference method to obtain solutions to the problems of a circular plate with a clamped edge and also with a simply supported edge. In the present paper, the postbuckling behavior of an annulus with free inner edge and clamped outer edge is investigated. The outer edge is subjected to a uniform inplane thrust. The von Kármán dynamic equations are employed. The postbuckling behavior of a clamped circular solid plate can be shown to be the limiting case of the annular problem. For axisymmetric deformations of annuli, the governing equations are two coupled nonlinear differential equations which form a nonlinear boundary value problem. With a suitable transformation, the governing equations yield a nonlinear eigenvalue problem. This problem is solved by considering the related initial-value problem,[6-8] which can be integrated readily. The results obtained in the present study reduce to those of the linear case when the amplitude parameter of buckling tends to zero.[9]

Received April 5, 1973; revision received July 18, 1973. The author expresses his sincere thanks to F. J. McCormick for his help and comments during the preparation of this paper.
Index category: Structural Stability.
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Mathematical Formulation

Consider an isotropic annulus, having outer radius a, inner radius b, thickness h, Young's modulus E, and Poisson ratio ν, subjected to a uniform compressive thrust q, at the outer edge. Denoting the transverse displacement of the unstrained midplane by w, the Airy stress function by ψ, and assuming axisymmetric deformations, the von Kármán equations[7,10] can be written as the following form

$$\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right]^2 w - \frac{12(1-\nu^2)}{Eh^3}\frac{1}{r}\frac{d}{dr}\left(\psi\frac{dw}{dr}\right) = 0 \tag{1a}$$

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} - \frac{\psi}{r^2} = -\frac{Eh}{2r}\left(\frac{dw}{dr}\right)^2 \tag{1b}$$

in which the relationships of the Airy stress function and the midplane thrusts are defined as

$$N_r = \int_{-h/2}^{h/2} \sigma_r\, dz = \frac{\psi}{r}, \qquad N_\theta = \int_{-h/2}^{h/2} \sigma_\theta\, dz = \frac{d\psi}{dr}$$

where N_r and N_θ denote the radial and circumferential midplane thrust, respectively.

For a plate with a clamped-movable outer edge and a free inner edge, the boundary conditions are:

a) at $r = a$

$$w = 0, \qquad dw/dr = 0, \qquad \psi = -q \tag{2a}$$

b) at $r = b$

$$\begin{aligned}(d^2w/dr^2)+(\nu/r)(dw/dr) &= 0\\ (d^3w/dr^3)+(1/r)(d^2w/dr^2)-(1/r^2)(dw/dr) &= 0\\ \psi &= 0\end{aligned} \tag{2b}$$

By introducing the dimensionless quantities

$$\xi = r/a, \qquad \chi = w/a \quad \text{and} \quad \phi = \psi/(Eha)$$

Equations (1) and (2) can be written as

$$\left[\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d}{d\xi}\right)\right]^2 \chi - 12(1-\nu^2)\left(\frac{a}{h}\right)^2 \frac{1}{\xi}\frac{d}{d\xi}\left(\phi\frac{d\chi}{d\xi}\right) = 0 \tag{3a}$$

$$\frac{d^2\phi}{d\xi^2}+\frac{1}{\xi}\frac{d\phi}{d\xi}-\frac{\phi}{\xi^2}=-\frac{1}{2\xi}\left(\frac{d\chi}{d\xi}\right)^2 \tag{3b}$$

and the boundary conditions

a) at $\xi = 1$

$$\chi = 0, \qquad \chi' = 0, \qquad \phi = -q/(Eha) = -p \tag{4a}$$

b) at $\xi = b/a = R$

$$\begin{aligned}\chi''+v/R\,\chi' &= 0\\ \chi'''+(1/R)\chi''-(1/R^2)\chi' &= 0\\ \phi &= 0\end{aligned} \tag{4b}$$

Equations (3) and (4) form a nonlinear boundary value problem for the determination of the axisymmetric equilibrium states of the clamped annulus. This boundary value problem depends upon the edge thrust p. Buckling takes place as soon as the prescribed edge thrust reaches a certain critical value, say, p_{cr}. In solving the buckling problem, Eqs. (3) and (4) are studied most conveniently by introducting the related nonlinear eigenvalue problem. Thus we introduce the transformation defined by

$$\phi(\xi)=\left(\frac{h}{a}\right)^2 v(\xi)-\frac{p}{1-R^2}\left(\xi-\frac{R^2}{\xi}\right)$$

and

$$u(\xi) = \chi(\xi)/\gamma$$

in which γ is a positive parameter.

Then Eqs. (3) can be reduced to the form

$$\frac{d^3u}{d\xi^3}+\frac{1}{\xi}\frac{d^2u}{d\xi^2}+\left[\frac{\lambda}{1-R^2}\left(1-\frac{R^2}{\xi^2}\right)-\frac{1}{\xi^2}\right]\frac{du}{d\xi}= 12(1-v^2)\frac{1}{\xi}\left(v\frac{du}{d\xi}\right) \tag{5a}$$

$$\frac{d^2v}{d\xi^2}+\frac{1}{\xi}\frac{dv}{d\xi}-\frac{v}{\xi^2}=-\frac{\alpha^2}{2\xi}\left(\frac{du}{d\xi}\right)^2 \tag{5b}$$

in which λ and α are nondimensional eigenvalue and amplitude parameters defined by

$$\lambda = 12(1-v^2)(a/h)^2 p \qquad \alpha = a\gamma/h$$

respectively. The boundary conditions, Eqs. (4), are reduced to

a) at $\xi = 1$

$$u = 0, \qquad du/d\xi = 0, \qquad v = 0 \tag{6a}$$

b) at $\xi = R$

$$(d^2u/d\xi^2)+(v/R)(du/d\xi) = 0, \qquad v = 0 \tag{6b}$$

Finally, a unique solution of Eqs. (5) and (6) is assured by introducing the normalization condition

$$u(R) = 1 \tag{6c}$$

By a limiting process, it can be shown that Eqs. (5) and (6) reduce to those given by Keller and Reiss[5] for the circular solid plate.

Analysis of the Nonlinear Eigenvalue Problem

Due to the nonlinearity in the eigenvalue problem, an analysis is proposed through the direct application of initial value methods. The field equations (5a) and (5b) can be written as a system of first-order nonlinear differential equations

$$d\bar{Y}/d\xi = \bar{H}(R \leqq \xi \leqq 1; \bar{Y}; \alpha, \lambda) \tag{7}$$

where

$$\begin{aligned}\bar{Y}(\xi) &= \{v, dv/d\xi, u, du/d\xi, d^2u/d\xi^2\}^T\\ &= \{y_1, y_2, y_3, y_4, y_5\}^T\end{aligned}$$

and $\bar{H}$ is the appropriate (5×1) vector function defined as

$$\bar{H}=\left\{y_2,\ -\frac{\alpha^2}{2\xi}y_4{}^2-\frac{1}{\xi}y_2+\frac{1}{\xi^2}y_1,\ y_4,\ y_5,\ 12(1-v^2)\frac{1}{\xi}y_1y_4-\frac{1}{\xi}y_5-\left[\frac{\lambda}{1-R^2}\left(1-\frac{R^2}{\xi^2}\right)-\frac{1}{\xi^2}\right]y_4\right\}^T$$

The boundary conditions (6a) and (6b) and the normalization condition (6c) can be written in the generalized forms

$$(M)\,\bar{Y}(R) = \{0, 1, 0\}^T \tag{8a}$$

$$(N)\,\bar{Y}(1) = \{0, 0, 0\}^T \tag{8b}$$

where

$$(M)=\begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&v/R&1\end{bmatrix}$$

and

$$(N)=\begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix}$$

The system of Eqs. (7) is conveniently studied by using the following related initial value problem:

$$d\bar{Z}/d\xi = \bar{H}(\xi; \bar{Z}; \alpha, \lambda) \tag{9a}$$

$$\begin{aligned}\bar{Z}(R) &= [0, \eta_1, 1, \eta_2, -(v/R)\eta_2]^T\\ &= [v, v', u, u', u'']^T_{\xi=R}\end{aligned} \tag{9b}$$

Among the components of the initial value vector (9b), η_i's are missing conditions; the remaining components are the given boundary conditions and the normalized condition at $\xi = R$. Hence a solution of the initial value problem, Eqs. (9), is symbolically indicated by

$$\begin{aligned}\bar{Z}(\xi) &\equiv \bar{Z}(\xi; \eta_1, \eta_2, \lambda; \alpha)\\ &= \bar{Z}(R)+\int_R^\xi \bar{H}(\xi; \bar{Z}; \eta_1, \eta_2, \lambda; \alpha)\,d\zeta\end{aligned}$$

Now, for a known value, α^0, we seek values of the parameters $(\eta_1, \eta_2, \lambda)$ such that the corresponding solution of Eqs. (9) satisfies the three end conditions (8b),

$$(N)\,\bar{Z}(\xi = 1; \eta_1, \eta_2, \lambda; \alpha^0) = 0 \tag{10}$$

It is now apparent that solving the eigenvalue problem [Eqs. (5) and (6)] is equivalent to finding a root of Eqs. (10). Thus for $\alpha = \alpha^0$,

$$\bar{Y}(\xi) = \bar{Z}(\xi; \eta_1{}^*, \eta_2{}^*, \lambda^*; \alpha^0)$$

is a solution to the eigenvalue problem [Eqs. (5) and (6)], where

$$\bar{S}^* = (\eta_1{}^*, \eta_2{}^*, \lambda^*)^T$$

is a root of

$$(N)\bar{Z}(\xi = 1; \bar{S}; \alpha^0) = 0 \tag{11}$$

To solve Eq. (11), we first choose the special parameter values $\alpha^0 = \eta_1{}^0 = 0$ and $\eta_2{}^0$. It follows that the first two equations of Eqs. (7) yield the results

$$\bar{y}_1(\xi) = \bar{y}_2(\xi) \equiv 0$$

The remaining equations of Eqs. (7) reduce to the basic equation of the buckling of an annular plate in the case of small deflection,

$$\frac{d^3u}{d\xi^3}+\frac{1}{\xi}\frac{d^2u}{d\xi^2}+\left[\frac{\lambda}{1-R^2}\left(1-\frac{R^2}{\xi^2}\right)-\frac{1}{\xi^2}\right]\frac{du}{d\xi}=0$$

which has been studied by Rozsa.[8] Solving the three equations (11) for the three unknowns, $\bar{S}$, can be accomplished by a direct application of Newton-Kantorovich method.[11] Starting from a special initial parameter, $\bar{S}^0 = (0, \eta_2{}^0, \lambda^0)$ and $\alpha^0 = 0$, the convergent sequence

$$\bar{S}^{(k+1)} = \bar{S}^{(k)}+\Delta\bar{S}^{(k)} \qquad k = 0, 1, 2, \ldots \tag{12}$$

is generated, provided the following linear corrector exists

$$\Delta\bar{S}^{(k)} = -[(N)J_1{}^{(k)}]^{-1}(N)\bar{Z}(1; \bar{S}^{(k)}; \alpha^0) \tag{13}$$

The Fréchet derivative, $J_1{}^{(k)}$, is defined as

$$J_1{}^{(k)} = \left.\frac{\partial\bar{Z}}{\partial\bar{S}}\right|^{(k)}_{\xi=1} \tag{14}$$

which can be constructed readily from the variational equations of the Eqs. (9) (Refs. 6 and 7).

The analysis of the post-buckling problem is completed, when

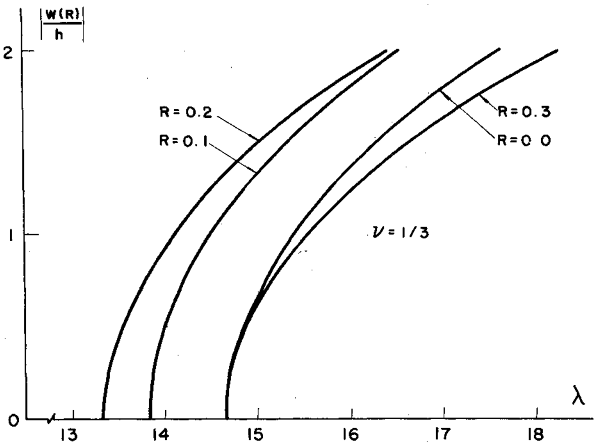

Fig. 1 Postbuckling loads and transverse displacement parameters for a circular plate.

the functional, $\bar{S} = \bar{S}(\alpha)$, is established. It can be achieved by the continuation method.[7,12] Having obtained a root, $\bar{S}^*$, corresponding to $\alpha = 0$, the function $\bar{S}(\alpha)$, can be determined by perturbing the value α in the following manner,

$$\alpha^{(k+1)} = \alpha^{(k)} + \Delta\alpha^{(k)} \qquad k = 0, 1, 2, \ldots, m$$

with $\alpha^{(0)} = 0$. The conditions sufficient to guarantee the existence, continuity, and uniqueness of the solution, $\bar{S}(\alpha)$, of Eq. (11) are given by Ficken.[12]

The nondimensional radial and circumferential bending stresses are given in terms of the negative transverse displacement, $u(\xi)$, by

$$\sigma_r{}^B a^2 h/6D = \alpha[d^2u/d\xi^2 + (v/\xi)\, du/d\xi] \tag{15a}$$

$$\sigma_\theta{}^B a^2 h/6D = \alpha[(1/\xi)\, du/d\xi + v\, d^2u/d\xi^2] \tag{15b}$$

respectively, and the nondimensional radial and circumferential membrane stresses are given in terms of the reduced stress function, $v(\xi)$, by

$$\frac{\sigma_r{}^M a^2 h}{D} = 12(1-v^2)\frac{v}{\xi} - \frac{\lambda}{1-R^2}\left(1 - \frac{R^2}{\xi^2}\right) \tag{16a}$$

$$\frac{\sigma_\theta{}^M a^2 h}{D} = 12(1-v^2)\frac{dv}{d\xi} - \frac{\lambda}{1-R^2}\left(1 + \frac{R^2}{\xi^2}\right) \tag{16b}$$

respectively.

With the aid of l'Hospital's rule, the stresses at $\xi = 0$ for a circular solid plate are found to be

$$\left(\frac{\sigma_r{}^B a^2 h}{6D}\right)_{\xi=0} = \left(\frac{\sigma_\theta{}^B a^2 h}{6D}\right)_{\xi=0} = \alpha(1+v)\left.\frac{d^2u}{d\xi^2}\right|_{\xi=0} \tag{17a}$$

$$\left(\frac{\sigma_r{}^M a^2 h}{D}\right)_{\xi=0} = \left(\frac{\sigma_\theta{}^M a^2 h}{D}\right)_{\xi=0} = 12(1-v^2)\left.\frac{dv}{d\xi}\right|_{\xi=0} - \lambda \tag{17b}$$

Numerical Computation

The analysis presented above suggests the use of a numerical integration technique. Thus, by integrating the initial value problem (9) using a fourth-order Runge-Kutta method, and performing the successive iterations by Newton-Kantorovich method, numerical solutions to the nonlinear eigenvalue problem are obtained. The procedure of numerical computation used in this study is given as follows:

The problem of linear buckling of an annulus is first investigated ($\alpha = 0$, $\eta_1 = v'(R) = 0$). For this case, the equation which governs the transverse displacement is linear, and the solution is obtained and agrees with Rozsa's result. Then, extracting the values of parameters for η_1, η_2 and λ from this linear solution, and setting $\Delta\alpha = 0.3$, the initial value problem (9) is integrated numerically with step size $\Delta\xi = \frac{1}{40}$ on $[R, 1]$. Iteration and integration are carried out until the norm of $(N)\bar{Z}(\xi = 1; \bar{S}; \alpha)$ satisfies the inequality

$$\sum^{3} \|(N)\bar{Z}(1; \bar{S}; \alpha)\| \leq 0.1 \times 10^{-5} \tag{18}$$

The problem of a post-buckling annulus has been solved for three different ratios of radii, i.e., $R = 0.3$, 0.2, and 0.1. The case of a solid circular plate is also investigated by taking the limit of Eqs. (5) and (6) as R approaches zero. Hence the Eqs. (5) and (6) can be rewritten as

$$\frac{d^2u'}{d\xi^2} + \frac{1}{\xi}\frac{du'}{d\xi} + \left(\lambda - \frac{1}{\xi^2}\right)u' = 12(1-v^2)\frac{1}{\xi}(vu') \tag{19a}$$

$$\frac{d^2v}{d\xi^2} + \frac{1}{\xi}\frac{dv}{d\xi} - \frac{v}{\xi^2} = -\frac{\alpha^2}{2\xi}\left(\frac{du}{d\xi}\right)^2 \tag{19b}$$

and the boundary conditions

a) at $\xi = 1$

$$u' = 0, \qquad v = 0 \tag{20a}$$

b) $\xi = 0$

$$du'/d\xi = 0, \qquad v = 0 \tag{20b}$$

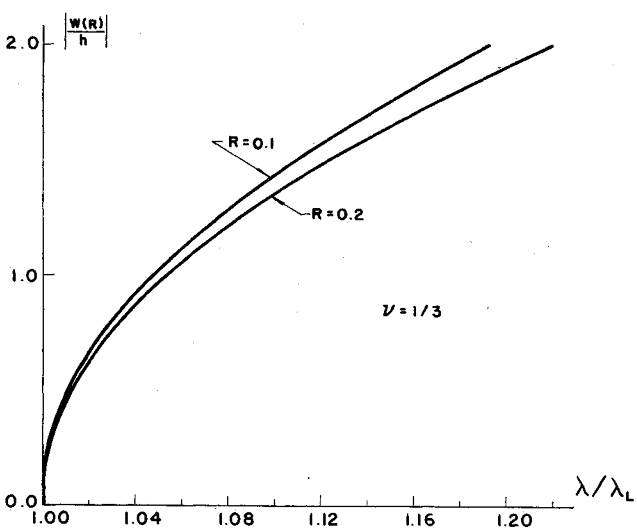

Fig. 2 Ratio of postbuckling load to linear buckling load and transverse displacement parameter.

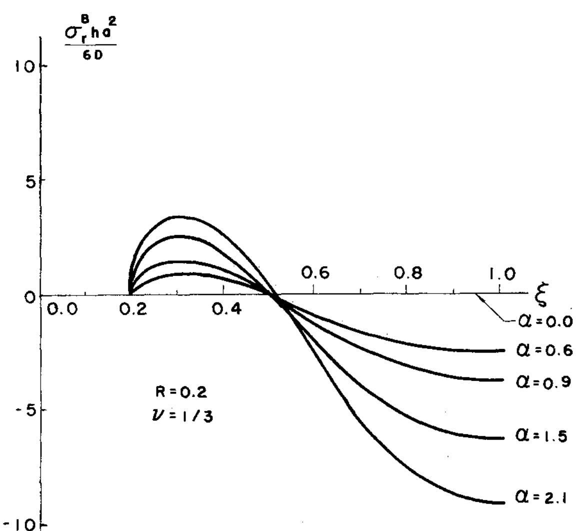

Fig. 3 The distribution of radial bending stresses.

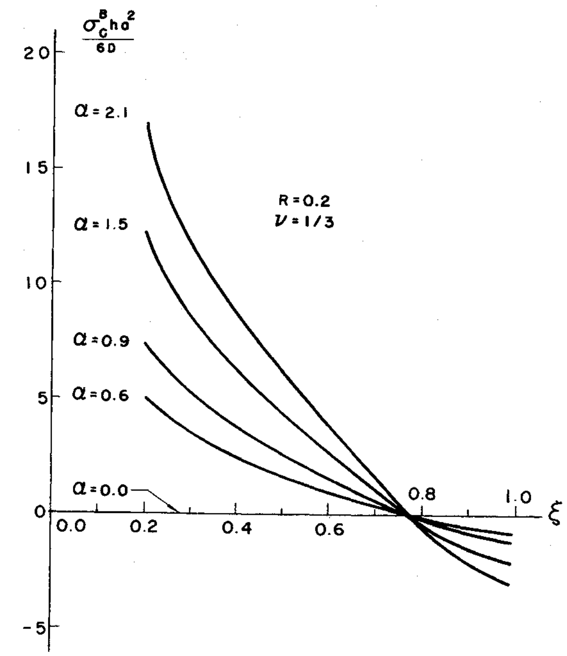

Fig. 4 The distribution of circumferential stresses.

in which

$$u' = du/d\xi$$

This system of nonlinear eigenvalue equations is in agreement with that given by Keller and Reiss[5] and is solved by applying the foregoing procedure.

Results and Discussion

In the present work attention was given mainly to an annulus with a clamped-movable outer edge and a free inner edge. In this paper, the ratios of radii, R, considered are 0.3, 0.2, and 0.1; Poisson's ratio is $\frac{1}{3}$. The result for a clamped circular solid plate was also obtained by a limiting process. Although only the clamped-free annulus is investigated in this paper, the generalized boundary conditions can be treated by this technique without further difficulty. The only modification needed is to change the matrices (M) and (N) in Eq. (8). All calculations were carried out on an IBM 360/50 computer. The linear buckling loads of

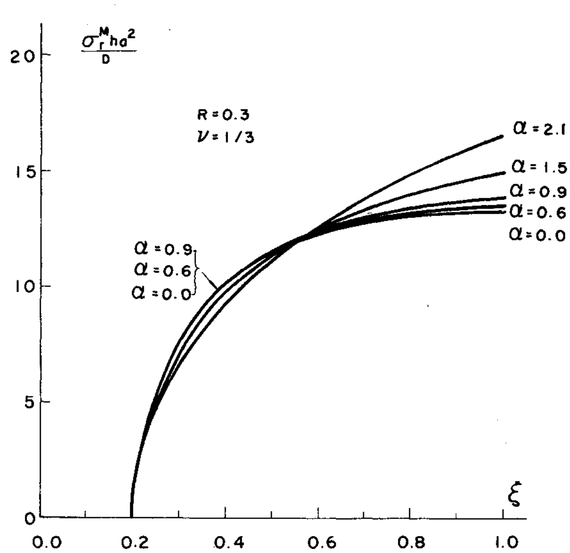

Fig. 5 The distribution of radial membrane stresses.

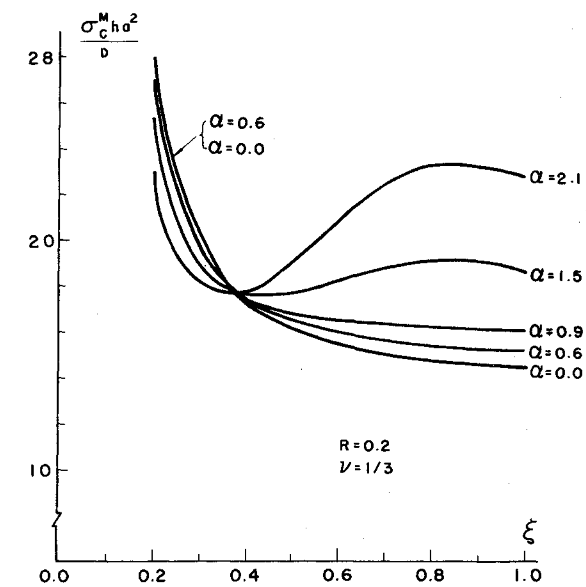

Fig. 6 The distribution of circumferential membrane stresses.

a thin elastic annulus are listed in Table 1 for various values of R. These loads agree with those obtained from the closed form solution.[9] The post-buckling curves for radii ratios of $R = 0.3$, 0.2, 0.1 and 0.0 are shown in Fig. 1. The corresponding values of the postbuckling loads are given in Table 2. As expected,

Table 1 Linear buckling loads of an annular plate with free inner edge and clamped outer edge

R	0.0	0.1	0.2	0.3
λ	14.682	13.848	13.393	14.691

Table 2 Postbuckling loads of an annular plate with free inner edge and clamped outer edge

	R			
α^2	$\lambda = 0.0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$
0.00	14.6820	13.8482	13.3930	14.6909
0.18	14.8096	13.9658	13.5254	14.8477
0.36	14.9373	14.0834	13.6579	15.0045
0.54	15.0651	14.2011	13.7903	15.1613
0.72	15.1930	14.3190	13.9228	15.3179
0.90	15.3211	14.4369	14.0553	15.4745
1.08	15.4492	14.5550	14.1878	15.6310
1.26	15.5776	14.6731	14.3204	15.7875
1.44	15.7060	14.7914	14.4529	15.9439
1.62	15.8345	14.9098	14.5855	16.1002
1.80	15.9632	15.0283	14.7181	16.2564
1.98	16.0920	15.1470	14.8507	16.4126
2.16	16.2209	15.2657	14.9833	16.5688
2.34	16.3499	15.3845	15.1160	16.7248
2.52	16.4791	15.5035	15.2486	16.8809
2.70	16.6083	15.6226	15.3813	17.0368
2.88	16.7377	15.7418	15.5140	17.1927
3.06	16.8672	15.8611	15.6467	17.3485
3.24	16.9969	15.9805	15.7795	17.5043
3.42	17.1266	16.1001	15.9123	17.6600
3.60	17.2565	16.2197	16.0451	17.8157
3.78	17.3865	16.3395	16.1779	17.9713
3.96	17.5167	16.4594	16.3107	18.1268
4.14		16.5794	16.4436	18.2823
4.32		16.6995	16.5765	18.4378

the buckling load increases as the amplitude of transverse deflection increases. The postbuckling of an annular plate is similar to that found for the oscillation of Duffing's hard spring system. In Fig. 2, the ratio of postbuckling load to linear buckling load is plotted as a function of amplitude parameter, which is defined as the ratio of the transverse deflection to the thickness. It shows the nonlinear effect of amplitude upon the buckling load parameter. The distributions of radial bending stress, circumferential bending stress, radial membrane stress and circumferential membrane stress are given in Figs. 3–6, respectively, for an annulus with $R = 0.2$. It is interesting to observe the rapid changes in stress levels near the edges. The rise in the circumferential stresses at the free edge, $\xi = R$, illustrate the occurrence of a stress concentration phenomenon. It is thus apparent that the circumferential stresses must be considered in any failure criteria for the plate design. In addition, it follows from Eqs. (5) and (6) that

$$\lambda(\alpha) = \lambda(-\alpha)$$

This result shows that the problem of postbuckling of a thin annulus is one of bifurcation. The value of λ for linear buckling is the one at which bifurcation can occur. In other words, one possible equilibrium configuration of the annulus is a state of uniform edge thrust in which the plate remains plane. However, this state is unstable when the edge thrust is larger than the linear buckling load and then two other states of equilibrium occur, as shown in Fig. 1.

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